Existence of weak solutions in lower order Sobolev space for a Camassa–Holm-type equation

2010 J. Phys. A: Math. Theor. 43 095205

(http://iopscience.iop.org/1751-8121/43/9/095205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 095205 (13pp) doi:10.1088/1751-8113/43/9/095205

Existence of weak solutions in lower order Sobolev space for a Camassa–Holm-type equation

Shaoyong Lai[1] **and Yonghong Wu**[2]

[1] Department of Applied Mathematics, Southwestern University of Finance and Economics, 610074, Chengdu, People's Republic of China
[2] Department of Mathematics and Statistics, Curtin University of Technology GPO Box 6845, Perth, Australia

E-mail: Laishaoy@swufe.edu.cn and y.wu@curtin.edu.au

Received 11 May 2009, in final form 18 January 2010
Published 15 February 2010
Online at stacks.iop.org/JPhysA/43/095205

Abstract

A generalized Camassa–Holm equation containing a nonlinear dissipative effect is investigated. The existence of the weak solution of the equation in lower order Sobolev space H^s with $1 < s \leqslant \frac{3}{2}$ is established by using the techniques of the pseudoparabolic regularization and some *a priori* estimates derived from the equation itself.

PACS numbers: 02.30.Ik, 47.35.Fg, 02.30.Hq
Mathematics Subject Classification: 35B10, 35Q35

1. Introduction

Camassa and Holm [3] proposed the following model equation for shallow water waves:

$$u_t - u_{xxt} + 3uu_x + 2ku_x = 2u_x u_{xx} + uu_{xxx}. \tag{1}$$

Alternative derivations of the equation as a model equation for water waves were made in [4] and [13]. It was shown in [3] that for all k, equation (1) is integrable. The inverse spectral or scattering approach was employed in [5] to study the equation. For $k = 0$, it is shown that equation (1) has traveling wave solutions of the form $c\,\mathrm{e}^{-|x-ct|}$ which, called peakons, capture an essential feature of the traveling waves of largest amplitude (see [21]). Equation (1) possesses bi-Hamiltonian structure and admits an infinite hierarchy of symmetries and conservation laws [1]. We should also address here that equation (1) is, in fact, a re-expression of geodesic flow (see Kouranbaeva [15] and Misiolek [19]), and this geometric interpretation leads to a proof that the least action principle holds (see [6]).

Recently, a lot of work has been carried out to study the properties and solutions of equation (1). Lennels [17] showed that, in addition to smooth solutions of (1), there exists a multitude of traveling waves with singularities such as peakons, cuspons, stumpons and composite waves. Wazwaz [22] obtained solitary wave solutions of (1) that include peakons, kinks, compactons, solitary pattern solutions and plane periodic solutions. The qualitative changes in the physical structures of the obtained solutions were shown in [22]. Hakkaev *et al* [12] investigated the existence and stability of periodic traveling wave solutions for the generalized Camassa–Holm and Benjamin–Bona–Mahony equations. The abstract results of Grillakis–Shatah–Strauss and the Floquet theory for periodic eigenvalue problems were employed in [12] to show orbital stability. Determinant formulas of N-soliton solutions of the continuous and semi-discrete Camassa–Holm equations are presented in Ohta *et al* [20] to generate multi-soliton, multi-cuspon and multi-soliton–cuspon solutions. Existence and uniqueness results for global weak solutions of (1) have been given in [9, 24, 25]. The sharpest results for the global existence and blowup solutions are found in [2]. Applying the Galerkin method and the Leray–Schauder fixed point theorem, Fu and Guo [10] investigated the existence and uniqueness of a time periodic solution for a class of viscous Camassa–Holm equation with periodic boundary conditions. Li and Olver [18] established the local well posedness in the Sobolev space $H^s(R)$ with $s > \frac{3}{2}$ for equation (1) and gave conditions on the initial data that led to finite time blowup of certain solutions. It was shown that the blowup occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time (see [5]). Lai and Xu [16] analyzed the compact and noncompact structures for two types of generalized Camassa–Holm–KP equations. For other methods to handle the problems related to the Camassa–Holm equation and functional spaces, the reader is referred to [7–9, 11, 26] and the references therein.

In this paper, we study the equation

$$u_t - u_{txx} + \partial_x f(u) = 2\alpha u_x u_{xx} + \alpha u u_{xxx} + \beta u^{2m} u_{xx}, \tag{2}$$

where $\alpha > 0$ and $\beta \geqslant 0$ are constants, and $f(u)$ is an $(n+1)$st-order polynomial with $f(u) = \sum_{j=1}^{n+1} a_j u^j$ and m is a natural number. Since the nonlinear term $\beta u^{2m} u_{xx}$ appears in equation (2), the conservation laws in previous work [18] for equations (1) lose their power to obtain some bounded estimates of the solution for equation (2). A new conservation law different from those presented in [18] will be established to prove the existence of weak solutions for equation (2) associated with the initial value $u_0(x) \in H^s(R)$. We should also address that all the generalized versions of the Camassa–Holm equation in previous works [16, 26] do not involve the nonlinear term $u^{2m} u_{xx}$. In addition, for an arbitrary positive Sobolev exponent, a lemma (see lemma 4.4 in section 4), which is similar to that presented in [1] where the Sobolev exponent is required to be greater than $\frac{3}{2}$, is established to prove the existence of solutions of the problem in the lower order Sobolev space H^s with $1 < s \leqslant \frac{3}{2}$.

2. Notation

The space of all infinitely differentiable functions $\phi(x,t)$ with compact support in $R \times [0, +\infty)$ is denoted by C_0^∞. We let p be any constant with $1 \leqslant p < +\infty$ and L^p be the space of all measurable functions $h(x, t)$ such that $\|h\|_{L^p}^p = \int_R |h(x,t)|^p\,\mathrm{d}x < \infty$. We define that L^∞ consists of all essentially bounded Lebesgue measurable functions h with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x\in R\setminus e} |h(x,t)|$. For any real number s, we let H^s denote the Sobolev space consisting of all tempered distributions h such that

$$\|h\|_{H^s} = \left(\int_R (1+|\xi|^2)^s |\hat{h}(\xi,t)|^2\,\mathrm{d}\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(\xi,t) = \int_R \mathrm{e}^{-\mathrm{i}x\xi} h(x,t)\,\mathrm{d}x$. Let $C([0,T]; H^s(R))$ denote the class of continuous functions from $[0,T]$ to $H^s(R)$ and $\Lambda = \left(1-\partial_x^2\right)^{\frac{1}{2}}$. Here we note that the norms $\|h\|_{L^p}^p$, $\|h\|_{L^\infty}$ and $\|h\|_{H^s}$ depend on time $t \in [0,\infty)$.

For simplicity, throughout this paper, we let c denote any positive constant which is independent of the parameter ε.

3. Well posedness of solutions for the regularized equation

In this section, we study the existence of solutions for equation (2) by considering its regularized equation with an initial condition in the form

$$\begin{cases} u_t - u_{txx} + \varepsilon u_{xxxxt} = -[f(u)]_x + 2\alpha u_x u_{xx} + \alpha u u_{xxx} + \beta u^{2m} u_{xx}, \\ u(x,0) = u_0(x), \end{cases} \tag{3}$$

where $0 < \varepsilon < \frac{1}{4}$. For problem (3), we have the local well-posedness theorem.

Theorem 3.1. *Let $u_0(x) \in H^s(R)$ with $s \geqslant 1$. Then there exists a unique solution $u(x,t) \in C([0,T]; H^s(R))$ where $T > 0$ depends on $\|u_0\|_{H^s(R)}$. If $s \geqslant 2$, the solution $u \in C([0,+\infty); H^s(R))$ exists for all time.*

Proof. We write the first equation in problem (3) in the form

$$\left(1 - \partial_x^2 + \varepsilon \partial_x^4\right) u_t = -(f(u))_x + \frac{\alpha}{2}\left(\partial_x^3 u^2 - \partial_x\left(u_x^2\right)\right) + \beta u^{2m} u_{xx}. \tag{4}$$

Let $D = \left(1 - \partial_x^2 + \varepsilon \partial_x^4\right)^{-1}$; then $D : H^s \to H^{s+4}$ is a bounded linear operator. Applying the operator D to both sides of equation (4) and then integrating the resultant equation with respect to t over the interval $(0,t)$ leads to

$$u(x,t) = u_0(x) + \int_0^t D\left[-(f(u))_x + \frac{\alpha}{2}\left(\partial_x^3 u^2 - \partial_x\left(u_x^2\right)\right) + \beta u^{2m} u_{xx}\right] \mathrm{d}t. \tag{5}$$

Suppose that both u and v are in the closed ball $B_{M_0}(0)$ of radius $M_0 > 1$ about the zero function in $C([0,T]; H^s(R))$ and A is the operator on the right-hand side of (5), for fixed $t \in [0,T]$; we get

$$\begin{aligned} &\left\| \int_0^t D\left[(-f(u))_x + \frac{\alpha}{2}\left(\partial_x^3 u^2 - \partial_x\left(u_x^2\right)\right) + \beta u^{2m} u_{xx}\right] \mathrm{d}t \right. \\ &\qquad \left. - \int_0^t D\left[(-f(v))_x + \frac{\alpha}{2}\left(\partial_x^3 v^2 - \partial_x\left(v_x^2\right)\right) + \beta v^{2m} v_{xx}\right] \mathrm{d}t \right\|_{H^s} \\ &\qquad \leqslant C_1 T \left(\sup_{0\leqslant t\leqslant T} \|u-v\|_{H^s} + \sup_{0\leqslant t\leqslant T} \sum_{j=1}^{n+1} \|u^j - v^j\|_{H^s} \right. \\ &\qquad\quad \left. + \sup_{0\leqslant t\leqslant T} \|u^2 - v^2\|_{H^s} + \sup_{0\leqslant t\leqslant T} \|D[u^{2m} u_{xx} - v^{2m} v_{xx}]\|_{H^s} \right), \end{aligned} \tag{6}$$

where C_1 may depend on ε. From the algebraic property of $H^{s_0}(R)$ with $s_0 > \frac{1}{2}$, we have

$$\begin{aligned} \|u^j - v^j\|_{H^s} &= \|(u-v)(u^{j-1} + u^{j-2}v + \cdots + uv^{j-2} + v^{j-1})\|_{H^s} \\ &\leqslant \|(u-v)\|_{H^s} \|(u^{j-1} + u^{j-2}v + \cdots + uv^{j-2} + v^{j-1})\|_{H^s} \\ &\leqslant c\|(u-v)\|_{H^s} \sum_{i=0}^{j-1} \|u\|_{H^s}^{j-1-i} \|v\|_{H^s}^{i}. \end{aligned} \tag{7}$$

Using $u^{2m}u_{xx} = \partial_x[u^{2m}u_x] - 2mu^{2m-1}(u_x)^2$ and $v^{2m}v_{xx} = \partial_x[v^{2m}v_x] - 2mv^{2m-1}(v_x)^2$, we get

$$\begin{aligned}\|D[u^{2m}u_{xx} - v^{2m}v_{xx}]\|_{H^s} &\leqslant \left\|D\partial_x[u^{2m}u_x - v^{2m}v_x]\right\|_{H^s} + c\left\|D\left[u^{2m-1}u_x^2 - v^{2m-1}v_x^2\right]\right\|_{H^s}\\ &\leqslant \|D[(u^{2m} - v^{2m})v_x + u^{2m}(u_x - v_x)]\|_{H^s}\\ &\quad + c\left\|D\left[u^{2m-1}\left(u_x^2 - v_x^2\right) + (u^{2m-1} - v^{2m-1})v_x^2\right]\right\|_{H^s}\\ &\leqslant c\big(\|(u^{2m} - v^{2m})v_x\|_{H^{s-2}} + \|u^{2m}(u_x - v_x)\|_{H^{s-2}}\\ &\quad + \left\|u^{2m-1}\left(u_x^2 - v_x^2\right)\right\|_{H^{s-2}} + \left\|(u^{2m-1} - v^{2m-1})v_x^2\right\|_{H^{s-2}}\big)\\ &\leqslant cM_0^{2m}\|u - v\|_{H^s},\end{aligned} \tag{8}$$

in which $s \geqslant 1$ is used. From (5)–(8), we obtain

$$\|Au - Av\|_{H^s} \leqslant \theta\|u - v\|_{H^s}, \tag{9}$$

where $\theta = \max\left(C_3TM_0^n, C_3TM_0^{2m}\right)$ and C_3 is independent of T. Choosing T sufficiently small so that $\theta < 1$, we know that A is a contraction. Applying the above inequality and (5) yields

$$\|Au\|_{H^s} \leqslant \|u_0\|_{H^s} + \theta\|u\|_{H^s}. \tag{10}$$

Choosing T sufficiently small so that $\theta M_0 + \|u_0\|_{H^s} < M_0$, we know that A maps $B_{M_0}(0)$ to itself. It follows from the contraction-mapping principle that the mapping A has a unique fixed point u in $B_{M_0}(0)$.

For $s \geqslant 2$, using the first equation of problem (3) derives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R \left(u^2 + u_x^2 + \varepsilon u_{xx}^2 + 2\beta(2m+1)\int_0^t u^{2m}u_x^2\,\mathrm{d}\tau\right)\mathrm{d}x = 0, \tag{11}$$

from which we have the conservation law

$$\int_R \left(u^2 + u_x^2 + \varepsilon u_{xx}^2 + 2\beta(2m+1)\int_0^t u^{2m}u_x^2\,\mathrm{d}\tau\right)\mathrm{d}x = \int_R \left(u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2\right)\mathrm{d}x. \tag{12}$$

The global existence result follows from the integral formula (5) and (12). □

4. Estimates of solutions

In this section, we will give the regularity estimates of solutions to the regularized initial value problem (3). Firstly, we cite two lemmas presented in [14].

Lemma 4.1 ([14]). *If $r > 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover*

$$\|uv\|_{H^r} \leqslant c(\|u\|_{L^\infty}\|v\|_{H^r} + \|u\|_{H^r}\|v\|_{L^\infty}),$$

where c is a constant depending only on r.

Lemma 4.2 ([14]). *Let $r > 0$. If $u \in H^r \bigcap W^{1,\infty}$ and $v \in H^{r-1} \bigcap L^\infty$, then*

$$\|[\Lambda^r, u]v\|_{L^2} \leqslant c(\|\partial_x u\|_{L^\infty}\|\Lambda^{r-1}v\|_{L^2} + \|\Lambda^r u\|_{L^2}\|v\|_{L^\infty}).$$

Lemma 4.3. *Let $s \geqslant 2$ and the function u(x, t) is a solution of the problem (3) and the initial data $u_0(x) \in H^s$. Then we have*

$$\begin{aligned}\|u\|_{H^1}^2 &\leqslant c\int_R \left(u^2 + u_x^2 + \varepsilon u_{xx}^2 + 2\beta(2m+1)\int_0^t u^{2m}u_x^2\,\mathrm{d}\tau\right)\mathrm{d}x\\ &= c\int_R \left(u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2\right)\mathrm{d}x.\end{aligned} \tag{13}$$

For $q \in (0, s-1]$, there is a constant c independent of ε such that

$$\int_R (\Lambda^{q+1}u)^2\,\mathrm{d}x \leqslant \int_R [(\Lambda^{q+1}u_0)^2) + \varepsilon(\Lambda^q u_{0xx})^2]\,\mathrm{d}x + c\int_0^t \|u_x\|_{L^\infty}\left(\|u\|_{H^q}^2 \sum_{j=1}^n \|u\|_{L^\infty}^{j-1} + \|u\|_{H^{q+1}}^2\right)\mathrm{d}\tau + c\int_0^t \|u\|_{H^{q+1}}^2\left(\|u\|_{L^\infty}^{2m} + \|u\|_{L^\infty}^{2m-1}\|u_x\|_{L^\infty} + \|u\|_{L^\infty}^{2m-2}\|u_x\|_{L^\infty}^2\right)\mathrm{d}\tau. \tag{14}$$

If $q \in [0, s-1]$, there is a constant c independent of ε such that

$$(1-2\varepsilon)\|u_t\|_{H^q} \leqslant c\|u\|_{H^{q+1}}\left(1 + \|u\|_{H^1}\sum_{j=1}^n \|u\|_{L^\infty}^{j-1} + \|u\|_{L^\infty}^{2m} + \|u\|_{L^\infty}^{2m-1}\|u_x\|_{L^\infty} + \|u\|_{L^\infty}^{2m-2}\|u_x\|_{L^\infty}^2\right). \tag{15}$$

Proof. Using $\|u\|_{H^1}^2 \leqslant c\int_R (u^2 + u_x^2)\,\mathrm{d}x$ and (12) derives (13).

Using $\partial_x^2 = -\Lambda^2 + 1$ and the Parseval equality gives rise to

$$\int_R \Lambda^q u \Lambda^q \partial_x^3(u^2)\,\mathrm{d}x = -2\int_R (\Lambda^{q+1}u)\Lambda^{q+1}(uu_x)\,\mathrm{d}x + 2\int_R (\Lambda^q u)\Lambda^q(uu_x)\,\mathrm{d}x. \tag{16}$$

For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of equation (4), noting (16) and integrating the new equation with respect to x by parts, we have the equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_R ((\Lambda^q u)^2 + (\Lambda^q u_x)^2 + \varepsilon(\Lambda^q u_{xx})^2)\,\mathrm{d}x\right] = -\int_R (\Lambda^q u)\Lambda^q[f(u)]_x\,\mathrm{d}x - \alpha\int_R (\Lambda^{q+1}u)\Lambda^{q+1}(uu_x)\,\mathrm{d}x + \frac{\alpha}{2}\int_R (\Lambda^q u_x)\Lambda^q\left(u_x^2\right)\mathrm{d}x + \alpha\int_R (\Lambda^q u)\Lambda^q(uu_x)\,\mathrm{d}x + \beta\int_R \Lambda^q(u)\Lambda^q(u^{2m}u_{xx})\,\mathrm{d}x. \tag{17}$$

We will estimate the terms on the right-hand side of (17) separately. For the first term, using integration by parts, the Cauchy–Schwartz inequality, and lemmas 4.1 and 4.2, for $j \geqslant 1$, we have

$$\int_R (\Lambda^q u)\Lambda^q(u^j u_x)\,\mathrm{d}x = \int_R (\Lambda^q u)[\Lambda^q(u^j u_x) - u^j\Lambda^q u_x]\,\mathrm{d}x + \int_R (\Lambda^q u)u^j\Lambda^q u_x\,\mathrm{d}x \leqslant c\|u\|_{H^q}\left(j\|u\|_{L^\infty}^{j-1}\|u_x\|_{L^\infty}\|u\|_{H^q} + \|u_x\|_{L^\infty}\|u\|_{L^\infty}^{j-1}\|u\|_{H^q}\right) + \frac{j}{2}\|u\|_{L^\infty}^{j-1}\|u_x\|_{L^\infty}\|\Lambda^q u\|_{L^2}^2 \leqslant c\|u\|_{H^q}^2\|u\|_{L^\infty}^{j-1}\|u_x\|_{L^\infty}, \tag{18}$$

where c only depends on j and q. Using the above estimate to the second term yields

$$\int_R (\Lambda^{q+1}u)\Lambda^{q+1}(uu_x)\,\mathrm{d}x \leqslant c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}. \tag{19}$$

For the third term, using lemma 4.1 yields

$$\begin{aligned}\int_R (\Lambda^q u_x)\Lambda^q\left(u_x^2\right) \mathrm{d}x &\leqslant \|\Lambda^q u_x\|_{L^2}\|\Lambda^q u_x^2\|_{L^2}\\ &\leqslant c\|u\|_{H^{q+1}}(\|u_x\|_{L^\infty}\|u_x\|_{H^q})\\ &\leqslant c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}.\end{aligned} \tag{20}$$

For the last term in (17), using lemma 4.1 results in

$$\begin{aligned}\left|\int_R (\Lambda^q u)\Lambda^q(u^{2m}u_{xx})\,\mathrm{d}x\right| &= \left|\int_R (\Lambda^q u)\Lambda^q\left[\partial_x(u^{2m}u_x) - 2mu^{2m-1}u_x^2\right]\mathrm{d}x\right|\\ &= \left|\int_R \left[(\Lambda^q u_x)\Lambda^q(u^{2m}u_x) + 2m\Lambda^q u\Lambda^q\left[u^{2m-1}u_x^2\right]\right]\mathrm{d}x\right|\\ &\leqslant c\left(\|u_x\|_{H^q}\|u^{2m}u_x\|_{H^q} + \|u\|_{H^q}\|u^{2m-1}u_x^2\|_{H^q}\right)\\ &\leqslant c\|u\|_{H^{q+1}}^2\left(\|u\|_{L^\infty}^{2m} + \|u\|_{L^\infty}^{2m-1}\|u_x\|_{L^\infty}\right.\\ &\quad\left. + \|u\|_{L^\infty}^{2m-2}\|u_x\|_{L^\infty}^2\right).\end{aligned} \tag{21}$$

It follows from (17)–(21) that

$$\begin{aligned}&\frac{1}{2}\int_R [(\Lambda^q u)^2 + (\Lambda^q u_x)^2 + \varepsilon(\Lambda^q u_{xx})^2]\,\mathrm{d}x\\ &\qquad - \frac{1}{2}\int_R [(\Lambda^q u_0)^2 + (\Lambda^q u_{0x})^2 + \varepsilon(\Lambda^q u_{0xx})^2]\,\mathrm{d}x\\ &\quad\leqslant c\int_0^t \|u_x\|_{L^\infty}\left(\|u\|_{H^q}^2\sum_{j=1}^n \|u\|_{L^\infty}^{j-1} + \|u\|_{H^{q+1}}^2\right)\mathrm{d}\tau\\ &\qquad + c\int_0^t \|u\|_{H^{q+1}}^2\left(\|u\|_{L^\infty}^{2m} + \|u\|_{L^\infty}^{2m-1}\|u_x\|_{L^\infty} + \|u\|_{L^\infty}^{2m-2}\|u_x\|_{L^\infty}^2\right)\mathrm{d}\tau,\end{aligned} \tag{22}$$

which results in inequalities (14).

Applying the operator $\left(1-\partial_x^2\right)^{-1}$ to multiply both sides of (4) yields the equation

$$(1-\varepsilon)u_t - \varepsilon u_{xxt} = \left(1-\partial_x^2\right)^{-1}\left[-\varepsilon u_t - [f(u)]_x + \frac{\alpha}{2}\left[\partial_x^3(u^2) - \partial_x\left(u_x^2\right)\right] + \beta u^{2m}u_{xx}\right]. \tag{23}$$

Using $(\Lambda^q u_t)\Lambda^q$ to multiply both sides of equation (23) for $q \in [0, s-1]$ and integrating the resultant equation by parts give rise to

$$\begin{aligned}&(1-\varepsilon)\int_R (\Lambda^q u_t)^2\,\mathrm{d}x + \varepsilon\int_R (\Lambda^q u_{xt})^2\,\mathrm{d}x\\ &\quad= \int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\left[-\varepsilon u_t - [f(u)]_x\right.\\ &\qquad\left. + \frac{\alpha}{2}\left[\partial_x^3(u^2) - \partial_x\left(u_x^2\right)\right] + \beta u^{2m}u_{xx}\right]\mathrm{d}x.\end{aligned} \tag{24}$$

On the right-hand side of equation (24), noting $f(u) = \sum_{j=1}^{n+1} a_j u^j$, we have

$$\left|\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(-\varepsilon u_t - a_1 u_x)\,\mathrm{d}x\right| \leqslant \varepsilon\|u_t\|_{H^q}^2 + |a_1|\|u_t\|_{H^q}\|u\|_{H^q} \tag{25}$$

and for $j \geqslant 2$

$$\begin{aligned}
&\left|\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(-a_j u^j-\frac{\alpha}{2}u_x^2\right)\mathrm{d}x\right| \leqslant \|u_t\|_{H^q}\\
&\quad\times\left(\int_R (1+\xi^2)^{q-1}\,\mathrm{d}\xi\left(\int_R\left[a_j\widehat{u^{j-1}}(\xi-\eta)\widehat{u}(\eta)+\frac{\alpha}{2}\widehat{u_x}(\xi-\eta)\widehat{u_x}(\eta)\right]\mathrm{d}\eta\right)^2\right)^{\frac{1}{2}}\\
&\quad\leqslant c\|u_t\|_{H^q}\left(\int_R\frac{c(\|u^{j-1}\|_{H^q}\|u\|_{L^2}+\|u_x\|_{L^2}\|u_x\|_{H^q})}{1+\xi^2}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&\quad\leqslant c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}^{j-2}+1\right),
\end{aligned}\tag{26}$$

in which we have used lemma 4.1. Since

$$\begin{aligned}
\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x^2(uu_x)\,\mathrm{d}x &= -\int_R(\Lambda^q u_t)\Lambda^q(uu_x)\,\mathrm{d}x\\
&\quad+\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_x)\,\mathrm{d}x,
\end{aligned}\tag{27}$$

it follows from Young's inequality $\|f\star g\|_{H^r}\leqslant\|f\|_{H^{p_1}}\|f\|_{H^{p_2}}$, $\frac{1}{p_1}+\frac{1}{p_2}=1+\frac{1}{r}$ and the inequality $(1+\xi^2)^l\leqslant[(1+(\xi-\eta)^2)^l+(1+\eta^2)^l]$, $l>0$, that

$$\begin{aligned}
&\left|\int_R(\Lambda^q u_t)\Lambda^q(uu_x)\,\mathrm{d}x\right|\leqslant c\|u_t\|_{H^q}\\
&\quad\times\left(\int_R c\left(\int_R\left[(1+(\xi-\eta)^2)^{\frac{q+1}{2}}+(1+\eta^2)^{\frac{q+1}{2}}\right]\hat{u}(\xi-\eta)\hat{u}(\eta)\,\mathrm{d}\eta\right)^2\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&\quad\leqslant c\|u_t\|_{H^q}\left(\|\widehat{\Lambda^{q+1}u}\star\widehat{u}\|_{L^2}+\|\widehat{u}\star\widehat{\Lambda^{q+1}u}\|_{L^2}\right)\\
&\quad\leqslant c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}
\end{aligned}\tag{28}$$

and

$$\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_x)\,\mathrm{d}x\right|\leqslant c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}.\tag{29}$$

Using the Cauchy–Schwartz inequality and lemma 4.1 yields

$$\begin{aligned}
&\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(u^{2m}u_{xx})\,\mathrm{d}x\right|\\
&\quad\leqslant c\|u_t\|_{H^q}\|\left(1-\partial_x^2\right)^{-1}\Lambda^q(u^{2m}u_{xx})\|_{L^2}\\
&\quad= c\|u_t\|_{H^q}\|\left(1-\partial_x^2\right)^{-1}\Lambda^q(\partial_x[u^{2m}u_x]-2mu^{2m-1}(u_x)^2)\|_{L^2}\\
&\quad\leqslant c\|u_t\|_{H^q}(\|u^{2m}u_x\|_{H^q}+\|u^{2m-1}(u_x)^2\|_{H^q})\\
&\quad\leqslant c\|u_t\|_{H^q}\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}^{2m}+\|u\|_{L^\infty}^{2m-1}\|u_x\|_{L^\infty}+\|u\|_{L^\infty}^{2m-2}\|u_x\|_{L^\infty}^2\right).
\end{aligned}\tag{30}$$

Applying (25)–(30) to (24) yields the inequality

$$\begin{aligned}
(1-2\varepsilon)\|u_t\|_{H^q}\leqslant c\|u\|_{H^{q+1}}\Bigg(&1+\|u\|_{H^1}\sum_{j=1}^{n}\|u\|_{L^\infty}^{j-1}+\|u\|_{L^\infty}^{2m}\\
&+\|u\|_{L^\infty}^{2m-1}\|u_x\|_{L^\infty}+\|u\|_{L^\infty}^{2m-2}\|u_x\|_{L^\infty}^2\Bigg)
\end{aligned}$$

for some constant $c>0$.

For a real number s with $s > 0$, suppose that the function $u_0(x)$ is in $H^s(R)$, and let $u_{\varepsilon 0}$ be the convolution $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$ of the function $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\phi(\varepsilon^{-\frac{1}{4}}x)$ and u_0 such that the Fourier transform $\widehat{\phi}$ of ϕ satisfies $\widehat{\phi} \in C_0^\infty$, $\widehat{\phi(\xi)} \geqslant 0$ and $\widehat{\phi(\xi)} = 1$ for any $\xi \in (-1, 1)$. Then we have $u_{\varepsilon 0}(x) \in C^\infty$. It follows from section 3 that for each ε satisfying $0 < \varepsilon < \frac{1}{4}$, the Cauchy problem

$$\begin{cases} u_t - u_{xxt} + \varepsilon u_{xxxxt} = -[(f(u)]_x + \dfrac{\alpha}{2}\left(\partial_x^3 u^2 - \partial_x\left(u_x^2\right)\right) + \beta u^{2m} u_{xx}, \\ u(x,0) = u_{\varepsilon 0}(x), \qquad x \in R, \end{cases} \tag{31}$$

has a unique solution $u_\varepsilon(x,t) \in C^\infty([0,\infty); H^\infty)$. For an arbitrary positive Sobolev exponent, we give the following lemma whose proof is similar to that of lemma 4.4 in [1] where the Sobolev exponent $s > \frac{3}{2}$ is required.

Lemma 4.4. *Under the above assumptions, the following estimates hold for any ε with $0 < \varepsilon < \frac{1}{4}$ and $s > 0$:*

$$\|u_{\varepsilon 0}\|_{H^q} \leqslant c, \qquad \text{if} \quad q \leqslant s, \tag{32}$$

$$\|u_{\varepsilon 0}\|_{H^q} \leqslant c\varepsilon^{\frac{s-q}{4}}, \qquad \text{if} \quad q > s, \tag{33}$$

$$\|u_{\varepsilon 0} - u_0\|_{H^q} \leqslant c\varepsilon^{\frac{s-q}{4}}, \qquad \text{if} \quad q \leqslant s, \tag{34}$$

$$\|u_{\varepsilon 0} - u_0\|_{H^s} = o(1), \tag{35}$$

where c is a constant independent of ε.

Proof. Using the Fourier transform leads to

$$\begin{aligned} \widehat{\phi_\varepsilon}(\xi) &= \int \mathrm{e}^{-\mathrm{i}x\xi}\varepsilon^{-\frac{1}{4}}\phi\left(\varepsilon^{-\frac{1}{4}}x\right)\mathrm{d}x = \int \mathrm{e}^{\mathrm{i}(\varepsilon^{-\frac{1}{4}}x)(\varepsilon^{\frac{1}{4}}\xi)}\phi\left(\varepsilon^{-\frac{1}{4}}x\right)d\left(\varepsilon^{-\frac{1}{4}}x\right) \\ &= \widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right). \end{aligned}$$

Furthermore, we have

$$\widehat{u_{\varepsilon 0}}(\xi) = \widehat{\phi_\varepsilon \star u_0} = \widehat{\phi_\varepsilon}(\xi)\widehat{u_0}(\xi) = \widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)\widehat{u_0}(\xi)$$

and

$$\begin{aligned} \|u_{\varepsilon 0}\|_{H^q}^2 &= \int_R (1+|\xi|^2)^q |\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)\widehat{u_0}(\xi)|^2\,\mathrm{d}\xi \\ &\leqslant \int_R \frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\left|\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)\right|^2 (1+|\xi|^2)^s|\widehat{u_0}(\xi)|^2\,\mathrm{d}\xi \\ &\leqslant \|u_0\|_{H^s}^2 \sup_{\xi\in R}\left[\frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\left|\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)\right|^2\right]. \end{aligned}$$

When $q \leqslant s$, we get

$$\sup_{\xi\in R}\left[\frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\left|\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)\right|^2\right] \leqslant \sup_{\xi\in R}\left|\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)\right|^2 \leqslant c,$$

which derives that inequality (32) holds. □

If $q > s$, it holds that

$$\sup_{\xi\in R}\left[\frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\left|\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)\right|^2\right] \leqslant \sup_{\varepsilon^{\frac{1}{4}}\xi=K\in R}\left[\frac{\left(1+\left|\varepsilon^{-\frac{1}{4}}K\right|^2\right)^q}{\left(1+\left|\varepsilon^{-\frac{1}{4}}K\right|^2\right)^s}|\widehat{\phi}(K)|^2\right]$$
$$\leqslant \varepsilon^{\frac{s-q}{2}}\sup_{K\in R}\left[\left(\varepsilon^{\frac{1}{2}}+|K|^2\right)^{q-s}|\widehat{\phi}(K)|^2\right]$$
$$\leqslant c\varepsilon^{\frac{s-q}{2}},$$

from which we know that (33) holds.

For $q \leqslant s$, we have

$$\|u_{\varepsilon 0}-u_0\|_{H^q}^2 = \int_R (1+|\xi|^2)^q\left|\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)\widehat{u_0}(\xi)-\widehat{u_0}(\xi)\right|^2 \mathrm{d}\xi$$
$$\leqslant \int_R \frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}(1+|\xi|^2)^s|\widehat{u_0}(\xi)|^2\left|\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)-1\right|^2 \mathrm{d}\xi$$
$$\leqslant \|u_0\|_{H^s}^2 \sup_{R\in R}\left[\frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\left|\widehat{\phi}\left(\varepsilon^{\frac{1}{4}}\xi\right)-1\right|^2\right]$$
$$\leqslant \|u_0\|_{H^s}^2\varepsilon^{\frac{s-q}{2}}\sup_{\varepsilon^{\frac{1}{4}}\xi=K\in R}\left[\left(\varepsilon^{\frac{1}{2}}+|K|^2\right)^{q-s}|\widehat{\phi}(K)-1|^2\right]$$
$$\leqslant c\varepsilon^{\frac{s-q}{2}},$$

which results in inequality (34). Expression (35) is a common result since $u_{\varepsilon 0}$ uniformly converges to u_0 in space $H^s(R)$ with $s > 0$.

Remark. For $s > 0$, using $\|u_\varepsilon\|_{L^\infty} \leqslant c\|u_\varepsilon\|_{H^{\frac{1}{2}+}} \leqslant c\|u_\varepsilon\|_{H^1}$,[3] $\|u_\varepsilon\|_{H^1}^2 \leqslant c\int_R\left(u_\varepsilon^2+u_{\varepsilon x}^2\right)\mathrm{d}x$, (13), (32) and (33), we know

$$\|u_\varepsilon\|_{L^\infty}^2 \leqslant c\|u_\varepsilon\|_{H^1}^2 \leqslant c\int_R\left(u_{\varepsilon 0}^2+u_{\varepsilon 0x}^2+\varepsilon u_{\varepsilon 0xx}^2\right)\mathrm{d}x$$
$$\leqslant c\left(\|u_{\varepsilon 0}\|_{H^1}^2+\varepsilon\|u_{\varepsilon 0}\|_{H^2}^2\right)$$
$$\leqslant c\left(c+c\varepsilon\times\varepsilon^{\frac{s-2}{2}}\right)$$
$$\leqslant c_0, \tag{36}$$

where c_0 is independent of ε.

5. Existence of solutions

Many partial differential equations have a smooth effect on their solutions. This effect admits more regularity than the corresponding initial data (see [1, 22]). The task of this section is to give a sufficient condition which guarantees that a solution of the generalized Camassa–Holm system (2) exists in the Sobolev space H^s with $1 < s \leqslant \frac{3}{2}$. Firstly, we use the regularized problem (3) to estimate norms of its solutions, showing that they are bounded. When the parameter ε is sufficiently small, the weak convergence of these solutions to a solution of the Camassa–Holm equation (2) with the initial value $u_0(x)$ is acquired.

Theorem 5.1. *If $u_0(x) \in H^s(R)$ with $s \in \left[1, \frac{3}{2}\right]$ such that $\|u_{0x}\|_{L^\infty} < \infty$. Let $u_{\varepsilon 0}$ be defined as in section 4. Then there exist constants $T > 0$ and c independent of ε such that the solution u_ε of problem (31) satisfies $\|u_{\varepsilon x}\|_{L^\infty} \leqslant c$.*

[3] $\|u_\varepsilon\|_{H^{\frac{1}{2}+}}$ means that there exists a sufficiently small δ such that $\|u_\varepsilon\|_{\frac{1}{2}+} = \|u_\varepsilon\|_{H^{\frac{1}{2}+\delta}}$.

Proof. Using the notation $u = u_\varepsilon$ and differentiating (23) with respect to x give rise to

$$(1-\varepsilon)u_{xt} - \varepsilon u_{xxxt} + \alpha u u_{xx} + \frac{\alpha}{2}u_x^2 = f(u) - \frac{\alpha}{2}u^2 - \Lambda^{-2}\left[\varepsilon u_{xt} + f(u) - \frac{\alpha}{2}(u^2 - u_x^2) - \beta\partial_x(u^{2m}u_{xx})\right]. \tag{37}$$

Letting $p > 0$ be an integer and multiplying the above equation by $(u_x)^{2p+1}$ and then integrating the resulting equation with respect to x yield the equality

$$\frac{1-\varepsilon}{2p+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_R (u_x)^{2p+2}\,\mathrm{d}x - \varepsilon\int_R (u_x)^{2p+1}u_{xxxt}\,\mathrm{d}x + \frac{p\alpha}{2p+2}\int_R (u_x)^{2p+3}\,\mathrm{d}x = \int_R (u_x)^{2p+1}\left(f(u) - \frac{\alpha}{2}u^2\right)\mathrm{d}x - \int_R (u_x)^{2p+1}\Lambda^{-2}\left[\varepsilon u_{xt} + f(u) - \frac{\alpha}{2}(u^2 - u_x^2) - \beta\partial_x(u^{2m}u_{xx})\right]\mathrm{d}x. \tag{38}$$

Applying Hölder's inequality, we get

$$\frac{1-\varepsilon}{2p+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_R (u_x)^{2p+2}\,\mathrm{d}x \leqslant \left\{\varepsilon\left(\int_R |u_{xxxt}|^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}} + \left(\int_R \left|f(u) - \frac{\alpha}{2}u^2\right|^{2p+2}\mathrm{d}x\right)^{\frac{1}{2p+2}} + \left(\int_R |G|^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}}\right\}\left(\int_R |u_x|^{2p+2}\,\mathrm{d}x\right)^{\frac{2p+1}{2p+2}} + \left|\frac{p\alpha}{2p+2}\right|\|u_x\|_{L^\infty}\int_R |u_x|^{2p+2}\,\mathrm{d}x \tag{39}$$

or

$$(1-\varepsilon)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_R (u_x)^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}} \leqslant \left\{\varepsilon\left(\int_R |u_{xxxt}|^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}} + \left(\int_R |f(u) - \frac{\alpha}{2}u^2|^{2p+2}\mathrm{d}x\right)^{\frac{1}{2p+2}} + \left(\int_R |G|^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}}\right\} + \left|\frac{p\alpha}{2p+2}\right|\|u_x\|_{L^\infty}\left(\int_R |u_x|^{2p+2}\,\mathrm{d}x\right)^{\frac{1}{2p+2}}, \tag{40}$$

where

$$G = \Lambda^{-2}\left[\varepsilon u_{xt} + f(u) - \frac{\alpha}{2}(u^2 - u_x^2) - \beta\partial_x(u^{2m}u_{xx})\right].$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \bigcap L^2$, integrating (40) with respect to t and taking the limit as $p \to \infty$ result in the estimate

$$(1-\varepsilon)\|u_x\|_{L^\infty} \leqslant (1-\varepsilon)\|u_{0x}\|_{L^\infty} + \int_0^t \left[\varepsilon\|u_{xxxt}\|_{L^\infty} + c\left(\|f(u) - \frac{\alpha}{2}u^2\|_{L^\infty} + \|G\|_{L^\infty}\right) + \frac{|\alpha|}{2}\|u_x\|_{L^\infty}^2\right]\mathrm{d}\tau. \tag{41}$$

For an arbitrary natural number n, using the algebraic property of $H^s(R)$ with $s > \frac{1}{2}$, inequalities (13) and (36) lead to

$$\|u^{n+1}\|_{L^\infty} \leqslant c\|u^{n+1}\|_{H^{\frac{1}{2}+}} \leqslant c\|u^{n+1}\|_{H^1} \leqslant c\|u\|_{H^1}^{n+1} \leqslant c, \tag{42}$$

from which we obtain $\|f(u)-\frac{\alpha}{2}u^2\|_{L^\infty}\leqslant c$. Applying $u^{2m}u_x=\partial_x[u^{2m}u_x]-2mu^{2m-1}u_x^2$ yields

$$\partial_x[u^{2m}u_x]=\partial_x^2[u^{2m}u_x]-2m\partial_x\big[u^{2m-1}u_x^2\big],$$

which results in

$$\begin{aligned}\|\Lambda^{-2}\partial_x(u^{2m}u_x)\|_{L^\infty}&\leqslant c\big(\|\Lambda^{-2}\partial_x^2(u^{2m}u_x)\|_{L^\infty}+\|\Lambda^{-2}\partial_x\big(u^{2m-1}u_x^2\big)\|_{L^\infty}\big)\\&\leqslant c\big(\|\Lambda^{-2}(1-\Lambda^2)(u^{2m}u_x)\|_{L^\infty}+\|\Lambda^{-2}\partial_x\big(u^{2m-1}u_x^2\big)\|_{L^\infty}\big)\\&\leqslant c\big(\|\Lambda^{-2}(u^{2m}u_x)\|_{H^{\frac{1}{2}+}}+\|u^{2m}u_x\|_{L^\infty}+\|\Lambda^{-2}\partial_x\big(u^{2m-1}u_x^2\big)\|_{H^{\frac{1}{2}+}}\big)\\&\leqslant c\big(\|u^{2m}u_x\|_{H^0}+\|u\|_{L^\infty}^{2m}\|u_x\|_{L^\infty}+\|u^{2m-1}u_x^2\|_{H^0}\big)\\&\leqslant c\leqslant\big(\|u\|_{L^\infty}^{2m}\|u\|_{H^1}+\|u\|_{L^\infty}^{2m}\|u_x\|_{L^\infty}+\|u\|_{L^\infty}^{2m-1}\|u_x\|_{L^\infty}^2\big)\\&\leqslant c\big(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big),\end{aligned}\tag{43}$$

from which we have

$$\begin{aligned}\|G\|_{L^\infty}&=\left\|\Lambda^{-2}\left[\varepsilon u_{xt}+f(u)-\frac{\alpha}{2}\big(u^2-u_x^2\big)-\beta\partial_x(u^{2m}u_{xx})\right]\right\|_{L^\infty}\\&\leqslant c\big(\|\Lambda^{-2}u_{xt}\|_{H^{\frac{1}{2}+}}+\|\Lambda^{-2}u_x^2\|_{H^{\frac{1}{2}+}}+\|\Lambda^{-2}\partial_x(u^{2m}u_{xx})\|_{L^\infty}\big)+c\\&\leqslant c\big(\|u_t\|_{L^2}+\|u\|_{H^1}^2+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big)+c\\&\leqslant c\big(\|u_t\|_{L^2}+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big)+c,\end{aligned}\tag{44}$$

where c is independent of ε. Using (15) and (44) results in

$$\int_0^t\|G\|_{L^\infty}\mathrm{d}\tau\leqslant c+c\int_0^t\big(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big)\,\mathrm{d}\tau.\tag{45}$$

Moreover, for any fixed $r\in\big(\frac{1}{2},1\big)$, there exists a constant c_r such that $\|u_{xxxt}\|_{L^\infty}\leqslant c_r\|u_{xxxt}\|_{H^r}\leqslant c_r\|u_t\|_{H^{r+3}}$. Using (15) and (36) yields

$$\|u_{xxxt}\|_{L^\infty}\leqslant c\|u\|_{H^{r+4}}\big(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big).\tag{46}$$

Choosing $q=r+3$, $u=u_\varepsilon$ in (14), we have

$$\begin{aligned}\|u\|_{H^{r+4}}^2\leqslant{}&\|u_0\|_{H^{r+4}}^2+\varepsilon\|u_0\|_{H^{r+5}}^2\\&+c\int_0^t\|u\|_{H^{r+4}}^2\big(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big)\,\mathrm{d}\tau.\end{aligned}\tag{47}$$

Making use of (33), (36) and (47) gives rise to

$$\|u\|_{H^{r+4}}^2\leqslant\varepsilon^{\frac{s-r-4}{2}}+\varepsilon\varepsilon^{\frac{s-r-5}{2}}+c\int_0^t\|u\|_{H^{r+4}}^2\big(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big)\,\mathrm{d}\tau,\tag{48}$$

from which we derive

$$\|u\|_{H^{r+4}}^2\leqslant\varepsilon^{\frac{s-r-4}{2}}\exp\left[c\int_0^t\big(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big)\,\mathrm{d}\tau\right].\tag{49}$$

It follows from (46) and (49) that

$$\begin{aligned}\|u_{xxxt}\|_{L^\infty}\leqslant{}&c\varepsilon^{\frac{s-r-4}{4}}\big(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big)\\&\times\exp\left[c\int_0^t\big(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\big)\,\mathrm{d}\tau\right].\end{aligned}\tag{50}$$

Applying (41), (45) (50) and $0 < \varepsilon < \frac{1}{4}$, we obtain

$$\|u_x\|_{L^\infty} \leqslant \|u_{0x}\|_{L^\infty} + c\int_0^t \left[\varepsilon^{\frac{s-r}{4}}\left(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\right)\right.$$
$$\left.\times \exp\left(c\int_0^\tau \left(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\right)\mathrm{d}\zeta\right) + 1 + \|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\right]\mathrm{d}\tau. \quad (51)$$

□

It follows from the contraction mapping principle that there is a $T > 0$ such that the equation

$$\|W\|_{L^\infty} = \|u_{0x}\|_{L^\infty} + c\int_0^t \left[\varepsilon^{\frac{s-r}{4}}\left(1+\|W\|_{L^\infty}+\|W\|_{L^\infty}^2\right)\right.$$
$$\left.\times \exp\left(c\int_0^\tau \left(1+\|W\|_{L^\infty}+\|W\|_{L^\infty}^2\right)\mathrm{d}\zeta\right) + 1 + \|W\|_{L^\infty}+\|W\|_{L^\infty}^2\right]\mathrm{d}\tau \quad (52)$$

has a unique solution $W \in C[0, T]$. Using the theorem presented on page 51 in [18] or theorem II in section I .1 in [23] derives that there are constants $T > 0$ and $c > 0$ independent of ε such that $\|u_x\|_{L^\infty} \leqslant W(t)$ for arbitrary $t \in [0, T]$, which leads to the conclusion of theorem 5.1.

Using theorem 5.1, lemma 4.4, (14), (15), notation $u_\varepsilon = u$ and Gronwall's inequality result in the inequalities

$$\|u_\varepsilon\|_{H^q} \leqslant \|u_\varepsilon\|_{H^{q+1}} \leqslant c\exp\left[c\int_0^t \left(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\right)\right]\mathrm{d}\tau \quad (53)$$

and

$$\|u_{\varepsilon t}\|_{H^r} \leqslant c\left(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\right)\exp\left[c\int_0^t \left(1+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\right)\right]\mathrm{d}\tau, \quad (54)$$

where $q \in (0, s]$, $r \in (0, s-1]$ and any $t \in [0, T]$. It follows from Aubin compactness theorem that there is a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_n}\}$, such that $\{u_{\varepsilon_n}\}$ and their temporal derivatives $\{u_{\varepsilon_n t}\}$ are weakly convergent to a function $u(x, t)$ and its derivative u_t in $L^2([0, T], H^s)$ and $L^2([0, T], H^{s-1})$, respectively. Moreover, for any real number $R_1 > 0$, $\{u_{\varepsilon_n}\}$ is convergent to the function u strongly in the space $L^2([0, T], H^q(-R_1, R_1))$ for $q \in [0, s)$ and $\{u_{\varepsilon_n t}\}$ converges to u_t strongly in the space $L^2([0, T], H^r(-R_1, R_1))$ for $r \in [0, s-1]$. Thus, we state the existence of a weak solution to equation (2) in the following.

Theorem 5.2. *Suppose that $u_0(x) \in H^s$ with $1 < s \leqslant \frac{3}{2}$ and $\|u_{0x}\|_{L^\infty} < \infty$. Then there exists a $T > 0$ such that equation (2) associated with the initial value $u_0(x)$ has a weak solution $u(x, t) \in L^2([0, T], H^s)$ in the sense of distribution and $u_x \in L^\infty([0, T] \times R)$.*

Proof. From theorem 5.1, we know that $\{u_{\varepsilon_n x}\}(\varepsilon_n \to 0)$ is bounded in the space L^∞. Thus, the sequences $\{u_{\varepsilon_n}\}$, $\{u_{\varepsilon_n x}\}$ and $\{u^2_{\varepsilon_n x}\}$ are weakly convergent to u, u_x and u_x^2 in $L^2([0, T], H^r(-R_1, R_1))$ for any $r \in [0, s-1)$, separately. Hence, u satisfies the equation

$$-\int_0^T\int_R u(g_t - g_{xxt})\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_R \left[\left(f(u)+\frac{\alpha}{2}u_x^2\right)g_x - \frac{\alpha}{2}u^2 g_{xxx} - \beta(u^{2m}u_x)g_x - 2m\beta u^{2m-1}u_x^2 g\right]\mathrm{d}x\,\mathrm{d}t \quad (55)$$

with $u(x, 0) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0, T] \times R)$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0, T] \times R)$ of X, there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0, T] \times R)$. As $\{u_{\varepsilon_n x}\}$ weakly converges to u_x in $L^2([0, T] \times R)$, it results that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0, T] \times R)$. □

Acknowledgments

The authors are very grateful to the reviewers for their helpful comments, which have led to a significant improvement of the paper. They also acknowledge the support of this work by the Key Project of Chinese Ministry of Education (109140).

References

[1] Bona J and Smith R 1975 *Phil. Trans. R. Soc.* A **278** 555–601
[2] Bressan A and Constantin A 2007 *Arch. Ration. Mech. Anal.* **183** 215–39
[3] Camassa R and Holm D 1993 *Phys. Rev. Lett.* **71** 1661–4
[4] Constantin A and Lannes D 2009 *Arch. Ration. Mech. Anal.* **193** 165–86
[5] Constantin A, Gerdjikov V S and Ivanov R I 2006 *Inverse Problems* **22** 2197–207
[6] Constantin A and Kolev B 2002 *J. Phys. A: Math. Gen.* **35** 51–79
[7] Coclite G M, Holden H and Karlsen K H 2005 *Discrete Contin. Dyn. Syst.* **13** 659–82
[8] Coclite G M, Holden H and Karlsen K H 2005 *SIAM J. Math. Anal.* **37** 1044–69
[9] Danchin R 2003 *J. Differ. Equ.* **192** 429–44
[10] Fu Y P and Guo B L 2006 *J. Math. Anal. Appl.* **313** 311–21
[11] Hakkaev S and Kirchev K 2005 *Commun. Part. Differ. Equ.* **30** 761–81
[12] Hakkaev S, Iliev I D and Kirchev K 2008 *J. Phys. A: Math. Theor.* **41** 085203
[13] Johnson R S 2002 *J. Fluid Mech.* **455** 63–82
[14] Kato T and Ponce G 1988 *Commun. Pure Appl. Math.* **41** 891–907
[15] Kouranbaeva S 1999 *J. Math. Phys.* **40** 857–68
[16] Lai S Y and Xu Y 2008 *Math. Comput. Modelling* **47** 1089–98
[17] Lenells J 2005 *J. Differ. Equ.* **217** 393–430
[18] Li Y and Olver P 2000 *J. Differ. Equ.* **162** 27–63
[19] Misiolek G 1998 *J. Geom. Phys.* **24** 203–8
[20] Ohta Y, Maruno K I and Feng B F 2008 *J. Phys. A: Math. Theor.* **41** 355205
[21] Toland J F 1996 *Topol. Methods Nonlin. Anal.* **7** 1–48
[22] Wazwaz A M 2005 *Appl. Math. Comput.* **165** 485–501
[23] Walter W 1970 *Differential and Integral Inequalities* (New York: Springer)
[24] Xin Z and Zhang P 2000 *Commun. Pure Appl. Math.* **53** 1411–33
[25] Xin Z and Zhang P 2002 *Commun. Partial Differ. Equ.* **27** 1815–44
[26] Zheng Y and Lai S Y 2008 *Phys. Lett.* A **372** 4141–43